

COMPLETELY BOUNDED KERNELS

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ABSTRACT. It is a classical result that scalar valued positive kernels have Kolmogorov decompositions. This has been extended in various ways, culminating in a version of the Kolmogorov decomposition for completely positive $\mathcal{L}(\mathcal{A}, \mathcal{B})$ valued kernels, \mathcal{A} and \mathcal{B} C^* -algebras [1]. The notion of a Kolmogorov decomposition has also been extended to not necessarily positive operator valued hermitian kernels in [3], where a condition for decomposability is shown to be that the kernel can be written as a difference of positive kernels. For $\mathcal{L}(\mathcal{A}, \mathcal{B})$ valued kernels, the appropriate analogue is that of a completely bounded kernel, which we define in both the hermitian and non-hermitian case. We show that the Schwartz boundedness condition of [3] implies the existence of a Kolmogorov decomposition for hermitian kernels, and that when \mathcal{A} is unital and \mathcal{B} is injective (much as in the Wittstock decomposition theorem), completely bounded kernels have Kolmogorov decompositions.

1. MOTIVATION

Decomposition properties of bounded maps play an important role in functional analysis. Some notable examples are

- The Hahn-Jordan decomposition of a signed real measure as the difference of two positive measures, and consequently, the decomposition of a complex measure in terms of a linear combination of four positive measures;
- The decomposition of a bounded linear functional as linear combination of four positive linear functionals;
- The decomposition of a bounded linear operator in terms of positive linear operators, with an analogous result true more generally in C^* -algebras;
- The Wittstock decomposition of a completely bounded linear map into $\mathcal{L}(\mathcal{H})$, \mathcal{H} a Hilbert space, in terms of completely positive maps (see, for example, [11] for definitions of complete positivity and complete boundedness). More generally, $\mathcal{L}(\mathcal{H})$ may be replaced by any injective C^* -algebra.

In each case one can express the norm of the object being decomposed in terms of the norms of the positive objects.

These notions may be generalised to kernels. Recall that for a fixed set X , a kernel on X is a map whose domain is $X \times X$. Depending on the example, the range might be anything from the complex numbers to the bounded linear maps between C^* -algebras. These have been studied even in the absence of an algebra structure and significant results have been obtained in [9]. The examples

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listed above can be viewed as kernels on one point sets. Kernels occur naturally in a number of settings, most notably in the study of reproducing kernels. Positive kernels (that is, those which when restricted to $F \times F$ are positive semidefinite, where $F \subset X$ finite) are particularly important. A fundamental property of complex valued positive kernels is that they possess a so-called Kolmogorov decomposition; that is, we can factor such a kernel k as $k(x, y) = f(x)^* f(y)$ for an appropriate f taking its values in some Hilbert space. This has been generalised numerous times, first to operator valued kernels, then by Gerard Murphy to kernels taking values in a C^* -algebra [10], and finally to $\mathcal{L}(\mathcal{A}, \mathcal{B})$ valued kernels, \mathcal{A} and \mathcal{B} C^* -algebras, by Barreto, Bhat, Liebscher and Skeide [1]. In the latter cases, the function in the factorisation takes its values in a Hilbert C^* -module or correspondence.

There is a natural way of defining hermitian kernels, and these play an important role in function theory. Obviously, the difference of positive kernels is hermitian, and it is natural to wonder if hermitian kernels can always be expressed as the difference of positive kernels. There are easy examples showing that this is in general not possible. However for operator valued hermitian kernels, there are characterisations of those with such a decomposition due to Laurent Schwartz [12] and Constantinescu and Gheondea [3], and in particular, such kernels also have a sort of Kolmogorov decomposition.

Our goal in this paper is to study the decomposition properties of $\mathcal{L}(\mathcal{A}, \mathcal{B})$ valued kernels, \mathcal{A} and \mathcal{B} unital C^* -algebras. We find that not only do the results of Constantinescu and Gheondea on hermitian kernels carry over to this setting, but that in keeping with the theory of completely bounded maps, there is a related notion of completely bounded kernel. Under the assumption that \mathcal{A} is unital and \mathcal{B} injective, completely bounded kernels are decomposable in a form generalising the Kolmogorov decomposition (much as in the case of the Wittstock theorems for completely bounded maps — see Theorem 6.1). In the hermitian case this gives an appealing alternative characterisation of decomposability to that of Schwartz, Constantinescu and Gheondea.

2. INTRODUCING AND CHARACTERISING THE MAIN CLASSES OF KERNELS

We begin by fixing some set X . As noted above, by a kernel on X we mean a map whose domain is $X \times X$. Let \mathcal{A}, \mathcal{B} denote C^* -algebras, which we assume to be unital, and let $\mathcal{L}(\mathcal{A}, \mathcal{B})$ be the space of bounded, linear maps from \mathcal{A} to \mathcal{B} . By $\mathbb{K}(\mathcal{A}, \mathcal{B})$ we mean the set of all kernels on X taking their values in $\mathcal{L}(\mathcal{A}, \mathcal{B})$. This set has an involution: if $k \in \mathbb{K}(\mathcal{A}, \mathcal{B})$ then we define a kernel k^* by

$$k^*(x, y)[a] = (k(y, x)[a^*])^*.$$

Notice that $(k^*)^* = k$. If $k = k^*$ then we call k **hermitian**. In the standard way we can decompose any kernel k as a linear combination of hermitian kernels, $k = \Re k + i \Im k$, where $\Re k = \frac{1}{2}(k + k^*)$ and $\Im k = \frac{1}{2i}(k - k^*)$.

Definition 2.1. A kernel k is **completely positive** if for any finite choice $(x_i, a_i, b_i)_{i=1}^n$ of elements of $X \times \mathcal{A} \times \mathcal{B}$,

$$\sum_{i,j=1}^n b_i^* k(x_i, x_j)[a_i^* a_j] b_j \geq 0.$$

It is not immediately obvious from this definition, but completely positive kernels are hermitian. We denote by $\mathbb{K}^+(\mathcal{A}, \mathcal{B})$ the set of all completely positive kernels on X taking their values in $\mathcal{L}(\mathcal{A}, \mathcal{B})$.

By $\mathbb{D}(\mathcal{A}, \mathcal{B})$ we mean the set of all kernels on X taking their values in $\mathcal{L}(\mathcal{A}, \mathcal{B})$ that can be expressed as the difference of two completely positive kernels. We call such kernels **decomposable**.

Since completely positive kernels are hermitian, the difference of two such kernels is obviously hermitian. The condition that a kernel be expressible as the difference of completely positive kernels was explored by Schwartz in [12] and by Constantinescu and Gheondea in [3] for operator-valued kernels. We will expand upon the characterisations of such kernels from [3].

These sets of kernels are nested, with $\mathbb{K}^+ \subset \mathbb{D} \subset \mathbb{K}$. They are all closed under addition and (real) scalar multiplication (positive real scalar multiplication in the case of \mathbb{K}^+ , which is a cone). The kernel spaces carry a natural partial order: we say $k_1 \leq k_2$ if $k_2 - k_1 \in \mathbb{K}^+$. We will observe the standard convention of writing $\mathbb{K}(\mathcal{B}, \mathcal{B})$ as $\mathbb{K}(\mathcal{B})$, and when we wish to emphasise the set X over which kernels are defined we shall use a subscript, as in $\mathbb{K}_X(\mathcal{A}, \mathcal{B})$. The following theorem is from [1].

Theorem 2.2. *Let $k \in \mathbb{K}(\mathcal{A}, \mathcal{B})$. Then the following are equivalent:*

- (i) $k \in \mathbb{K}^+(\mathcal{A}, \mathcal{B})$.
- (ii) *For any finite choice x_1, x_2, \dots, x_n of elements from X the (entrywise) map*

$$(k(x_i, x_j)) : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$$

is completely positive.

- (iii) *The kernel k can be decomposed in the following sense: there exists an $(\mathcal{A}, \mathcal{B})$ -correspondence E_k and a map $\iota : X \rightarrow E_k$ such that for all choices of x, y, a we have*

$$k(x, y)a = \langle a \cdot \iota(y), \iota(x) \rangle_{E_k}.$$

We denote $\iota(x)$ by k_x . (Note that in [1] inner products are linear in the second argument, while here we follow the convention that they are linear in the first.)

The pair (E_k, ι) is known as a Kolmogorov decomposition. With a view to applying that terminology more widely, we call this a **positive Kolmogorov decomposition**. For more characterisations of completely positive kernels and a proof of the above result see [1, Lemma 3.2.1, Theorem 3.2.3].

Theorem 2.3. *Let $k \in \mathbb{K}(\mathcal{A}, \mathcal{B})$. Then the following are equivalent:*

- (i) $k \in \mathbb{D}(\mathcal{A}, \mathcal{B})$.
- (ii) *There exists a completely positive kernel L such that $k \leq L$.*
- (iii) *There exists a completely positive kernel L such that $-L \leq k \leq L$.*
- (iv) *There exists an $(\mathcal{A}, \mathcal{B})$ -correspondence E_k , a self-adjoint contractive left \mathcal{A} -module map $J : E_k \rightarrow E_k$, and a map $\iota : X \rightarrow E_k$ such that for all choices of x, y, a we have*

$$k(x, y)a = \langle a \cdot \iota(y), J\iota(x) \rangle_{E_k}.$$

- (v) *There exists an $(\mathcal{A}, \mathcal{B})$ -correspondence E_k , a self-adjoint contractive left \mathcal{A} -module map $J : E_k \rightarrow E_k$ such that $J^2 = I_{E_k}$, and a map $\iota : X \rightarrow E_k$ such that for all choices of x, y, a we have*

$$k(x, y)a = \langle a \cdot \iota(y), J\iota(x) \rangle_{E_k}.$$

As before, we denote $\iota(x)$ by k_x .

Proof of Theorem 2.3. (i) \Rightarrow (ii): Let $k = k_1 - k_2$. Then

$$k + k_1 + k_2 = k + k_1 + (k_1 - k) = 2k_1 \geq 0.$$

Thus $k \geq -(k_1 + k_2)$. Similarly

$$(k_1 + k_2) - k = (k + k_2) + k_2 - k = 2k_2 \geq 0$$

so $k_1 + k_2 \geq k$. Since $k_1 + k_2$ is completely positive, set $L = k_1 + k_2$ to obtain the result.

(ii) \Rightarrow (iii): Immediate.

(iii) \Rightarrow (i): Let $k \leq L$. Then $k = L - (L - k)$ is the difference of completely positive kernels.

(i) \Rightarrow (v): This is a ‘Kreĭn-module’ construction analogous to constructing decompositions of completely positive kernels. Let $k = k_1 - k_2$ and let E_1 and E_2 be the correspondences in the decompositions of the completely positive kernels k_1 and k_2 respectively. Define $E_k : E_1 \oplus E_2$ and $k_x = k_{1,x} \oplus k_{2,x}$ and $J : E_k \rightarrow E_k$ by $J(e_1 \oplus e_2) = e_1 \oplus -e_2$. Then

$$\begin{aligned} k(x, y)[a] &= k_1(x, y)[a] - k_2(x, y)[a] = \langle a \cdot k_{1,y}, k_{1,x} \rangle - \langle a \cdot k_{2,y}, k_{2,x} \rangle \\ &= \langle a \cdot (k_{1,y} \oplus k_{2,y}), J(k_{1,x} \oplus k_{2,x}) \rangle \\ &= \langle a \cdot k_y, k_x \rangle. \end{aligned}$$

Notice that J is self-adjoint, a left \mathcal{A} -module map, and has $J^2 = I$. This completes the construction of the decomposition of k .

(v) \Rightarrow (iv): Immediate.

(iv) \Rightarrow (i): Since J is a self-adjoint element of the C^* -algebra $\mathcal{L}^a(E_k)$ it can be expressed as the difference of two positive elements, say $J = J_1 - J_2$. Since J is a left \mathcal{A} -module map, each of J_1, J_2 must also be one. Then we have

$$\begin{aligned} k(x, y)[a] &= \langle a \cdot k_y, J k_x \rangle \\ &= \langle a \cdot k_y, J_1 k_x - J_2 k_x \rangle \\ &= \langle a \cdot k_y, J_1 k_x \rangle - \langle a \cdot k_y, J_2 k_x \rangle \\ &= \langle J_1^{1/2}(a \cdot k_y), J_1^{1/2} k_x \rangle - \langle J_2^{1/2}(a \cdot k_y), J_2^{1/2} k_x \rangle \\ &= \langle a J_1^{1/2} k_y, J_1^{1/2} k_x \rangle - \langle a J_2^{1/2} k_y, J_2^{1/2} k_x \rangle, \end{aligned}$$

which gives decompositions of two completely positive kernels whose difference is k . \square

We are motivated by Definition 2.1 of completely positive kernels to attempt the following generalisation:

Definition 2.4. Let $k : X \times X \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{B})$. If, given any finite subset $F = \{x_1, x_2, \dots, x_n\}$ of X , the map

$$(k(x_i, x_j)) : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$$

is a completely bounded map, then we call k a **completely bounded kernel**.

Wittstock’s decomposition theorem [11, Theorem 8.5] tells us that completely bounded maps $\mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ can be decomposed as the difference of two completely positive maps with the norm of the sum of the completely positive maps equal to that of the completely bounded map. A partial converse to this statement was proved by Haagerup [4]: if the conclusion of Wittstock’s theorem holds and \mathcal{B} is a von Neumann algebra, then it is necessarily injective. Smith and Williams [13] and Huruya [6] found similar results for \mathcal{B} a nuclear and separable C^* -algebra, respectively, though the characterisation in the nuclear case is more complex. Huruya and Tomiyama also found that if one relaxes the norm constraint, there are examples of nonseparable, non-injective C^* -algebras for which every completely bounded map is the difference of completely positive maps [7].

We intend to explore decomposability in the setting of kernels. It is clear that any completely bounded map is a linear combination of hermitian completely bounded maps. Furthermore, the difference of two completely positive kernels is always completely bounded and hermitian. We therefore seek conditions under which completely bounded hermitian kernels can be expressed as the difference of completely positive kernels.

3. KOLMOGOROV DECOMPOSITIONS

Definition 3.1. A kernel $k \in \mathbb{K}(\mathcal{A}, \mathcal{B})$ has a **Kolmogorov decomposition** if there exists a triple (E_k, J, ι) such that E_k is an $(\mathcal{A}, \mathcal{B})$ -correspondence, $J \in \mathcal{L}^a(E_k)$ is a contractive left \mathcal{A} -module map and $\iota : X \rightarrow E_k$ such that for all choices of x, y, a we have:

$$k(x, y)[a] = \langle a \cdot \iota(y), J(\iota(x)) \rangle.$$

Suppose k has a Kolmogorov decomposition. Then

$$\begin{aligned} k^*(x, y)[a] &= (k(y, x)[a^*])^* \\ &= \langle a^* \iota(x), J \iota(y) \rangle^* \\ &= \langle J \iota(y), a^* \iota(x) \rangle \\ &= \langle a \iota(y), J^* \iota(x) \rangle. \end{aligned}$$

From this we see that a kernel with a Kolmogorov decomposition is hermitian if and only if J is self-adjoint. Where J is self-adjoint (or positive) we shall say k has a **self-adjoint (or positive) Kolmogorov decomposition**. We now know that:

- A kernel k is completely positive if and only if it has a positive Kolmogorov decomposition (since $\langle a \iota(y), J \iota(x) \rangle = \langle a J^{1/2} \iota(y), J^{1/2} \iota(x) \rangle = \langle a \tilde{\iota}(y), \tilde{\iota}(x) \rangle$). Consequently, our current meaning for positive Kolmogorov decomposition is precisely that expressed in Theorem 2.2.
- A kernel k is the difference of two completely positive kernels if and only if it has a self-adjoint Kolmogorov decomposition. This was the content of Theorem 2.3.
- If k has a self-adjoint Kolmogorov decomposition then we can assume that the operator J is unitary. We saw in Theorem 2.3 that we can take $J^2 = I$.

We make use of these facts in proving the next lemma, a generalisation of [3, Theorem 4.4]. Compare also with Theorem 8.3 of [11].

Lemma 3.2. Let $k \in \mathbb{K}(\mathcal{A}, \mathcal{B})$. Then k has a Kolmogorov decomposition if and only if there exist completely positive kernels L_1 and L_2 such that

$$(x, y) \mapsto \begin{pmatrix} L_1(x, y) & k(x, y) \\ k^*(x, y) & L_2(x, y) \end{pmatrix} : \mathcal{A} \rightarrow M_2(\mathcal{B})$$

is a completely positive kernel.

Proof. If $k(x, y)[a] = \langle a \cdot k_y, J k_x \rangle$ then we take

$$L_1(x, y)[a] = L_2(x, y)[a] = \langle a \cdot k_y, k_x \rangle$$

which gives us

$$\begin{pmatrix} L_1(x, y)[a] & k(x, y)[a] \\ k^*(x, y)[a] & L_2(x, y)[a] \end{pmatrix} = \begin{pmatrix} \langle a \cdot k_y, k_x \rangle & \langle a \cdot k_y, J k_x \rangle \\ \langle a \cdot k_y, J^* k_x \rangle & \langle a \cdot k_y, k_x \rangle \end{pmatrix}$$

where we have used $k^*(x, y)[a] = \langle a k_y, J^* k_x \rangle$. View $M_2(E_k)$ as an $M_2(\mathcal{B})$ -module. The left \mathcal{A} -action is defined by embedding \mathcal{A} in $\mathcal{L}^a(E_k)$, which can in turn be identified with the diagonal of $M_2(\mathcal{L}^a(E_k))$, which is completely isometrically isomorphic to $\mathcal{L}^a(M_2(E_k))$. Notice that

$$\begin{pmatrix} I & J^* \\ J & I \end{pmatrix}$$

is then a positive element of $\mathcal{L}^a(M_2(E_k))$ and commutes with the left \mathcal{A} -action. Combining this with the above give us

$$\begin{pmatrix} L_1(x, y)[a] & k(x, y)[a] \\ k^*(x, y)[a] & L_2(x, y)[a] \end{pmatrix} = \left\langle a \cdot \begin{pmatrix} k_y & 0 \\ 0 & k_x \end{pmatrix}, \begin{pmatrix} I & J^* \\ J & I \end{pmatrix} \begin{pmatrix} k_x & 0 \\ 0 & k_x \end{pmatrix} \right\rangle$$

which is a positive Kolmogorov decomposition. Thus

$$\begin{pmatrix} L_1 & k \\ k^* & L_2 \end{pmatrix}$$

is a completely positive kernel.

Conversely suppose that the matrix of kernels is completely positive. Then conjugation by an element of $M_2(\mathcal{B})$ preserves complete positivity. In particular

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^* \begin{pmatrix} L_1 & k \\ k^* & L_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} L_1 + L_2 + k + k^* & L_1 - L_2 - (k - k^*) \\ L_1 - L_2 + (k - k^*) & L_1 + L_2 - (k + k^*) \end{pmatrix}$$

is a completely positive kernel. The entries on the diagonal must be completely positive, from which we deduce that

$$-\frac{1}{2}(L_1 + L_2) \leq \frac{1}{2}(k + k^*) \leq \frac{1}{2}(L_1 + L_2).$$

Similarly, conjugation by the $M_2(\mathcal{B})$ element

$$\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$

tells us that

$$-\frac{1}{2}(L_1 + L_2) \leq \frac{-i}{2}(k - k^*) \leq \frac{1}{2}(L_1 + L_2).$$

By Theorem 2.3, the kernels $K_1 = \frac{1}{2}(k + k^*)$ and $K_2 = \frac{-i}{2}(k - k^*)$ have hermitian Kolmogorov decompositions: for $i = 1, 2$ there exist modules E_i , self-adjoint operators J_i on these spaces, and maps $X \rightarrow E_i : x \mapsto k_i(x)$ such that

$$K_i(x, y)[a] = \langle a \cdot k_i(y), J_i k_i(x) \rangle_{E_i}.$$

Furthermore, $k = K_1 + i K_2$. We conclude that k has a Kolmogorov decomposition, since

$$\begin{aligned} k(x, y)[a] &= \langle a \cdot k_1(y), J_1 k_1(x) \rangle + i \langle a \cdot k_2(y), J_2 k_2(x) \rangle \\ &= \langle a \cdot (k_1(y) \oplus k_2(y)), (J_1 \oplus -i J_2)(k_1(x) \oplus k_2(x)) \rangle_{E_1 \oplus E_2}. \end{aligned} \quad \square$$

If $\varphi_1, \varphi_2, \varphi : \mathcal{A} \rightarrow \mathcal{B}$ and we define

$$\Phi : M_2(\mathcal{A}) \rightarrow M_2(\mathcal{B}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \varphi_1(a) & \varphi(b) \\ \varphi^*(c) & \varphi_2(d) \end{pmatrix} \quad \Psi : \mathcal{A} \rightarrow M_2(\mathcal{B}) : a \mapsto \begin{pmatrix} \varphi_1(a) & \varphi(a) \\ \varphi^*(a) & \varphi_2(a) \end{pmatrix}$$

then it is a result due to Haagerup that Φ is completely positive if and only if Ψ is completely positive. Haagerup discusses this in [4], though this presentation is due to [11]. If we then allow a 2×2 matrix of kernels to act (at each point) as a Schur product

$$\begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}(x, y) \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{pmatrix} k_1(x, y)[a_{1,1}] & k_2(x, y)[a_{1,2}] \\ k_3(x, y)[a_{2,1}] & k_4(x, y)[a_{2,2}] \end{pmatrix}$$

then we obtain the following:

Corollary 3.3. *Let $k \in \mathbb{K}(\mathcal{A}, \mathcal{B})$. Then k has a Kolmogorov decomposition if and only if there exist completely positive kernels L_1 and L_2 such that*

$$(x, y) \mapsto \begin{pmatrix} L_1(x, y) & k(x, y) \\ k^*(x, y) & L_2(x, y) \end{pmatrix} : M_2(\mathcal{A}) \rightarrow M_2(\mathcal{B})$$

is a completely positive kernel. In this case, k is a completely bounded kernel and can be decomposed as a linear combination of at most four completely positive kernels.

The last statement of the corollary follows by decomposing J in the Kolmogorov decomposition of k into a linear combination of at most four positive maps and applying Theorem 2.3.

4. AN APPLICATION OF THE OFF-DIAGONAL METHOD

The study of the decomposability of a completely bounded map φ is related to the problem of completely positive completion of a 2×2 matrix with φ and φ^* in the off-diagonal positions, as noted by Haagerup [4]. Our goal is to use this relationship to show that any completely bounded kernel (into an appropriate space) has a Kolmogorov decomposition. Now, let $k \in \mathbb{K}_X(\mathcal{A}, \mathcal{B})$ and consider the following six statements:

- (i) There exist $L_1, L_2 \in \mathbb{K}_X^+(\mathcal{A}, \mathcal{B})$ such that

$$\begin{pmatrix} L_1 & k \\ k^* & L_2 \end{pmatrix} \in \mathbb{K}_X^+(M_2(\mathcal{A}), M_2(\mathcal{B})).$$

- (ii) There exist $L_1, L_2 \in \mathbb{K}_X^+(\mathcal{A}, \mathcal{B})$ such that for any finite subset $\{x_1, x_2, \dots, x_n\}$ of X the map

$$\left(\begin{pmatrix} L_1(x_i, x_j) & k(x_i, x_j) \\ k^*(x_i, x_j) & L_2(x_i, x_j) \end{pmatrix} \right)_{i,j=1}^n : M_n(M_2(\mathcal{A})) \rightarrow M_n(M_2(\mathcal{B}))$$

is completely positive.

- (iii) There exist $L_1, L_2 \in \mathbb{K}_X^+(\mathcal{A}, \mathcal{B})$ such that for any finite subset $\{x_1, x_2, \dots, x_n\}$ of X the map

$$\left(\begin{pmatrix} (L_1(x_i, x_j))_{i,j=1}^n & (k(x_i, x_j))_{i,j=1}^n \\ (k^*(x_i, x_j))_{i,j=1}^n & (L_2(x_i, x_j))_{i,j=1}^n \end{pmatrix} : M_2(M_n(\mathcal{A})) \rightarrow M_2(M_n(\mathcal{B})) \right)$$

is completely positive.

- (iv) Given any finite set $F = \{x_1, x_2, \dots, x_n\}$ of X there exist $L_1, L_2 \in \mathbb{K}_F^+(\mathcal{A}, \mathcal{B})$ such that the map

$$\left(\begin{pmatrix} L_1(x_i, x_j) & k(x_i, x_j) \\ k^*(x_i, x_j) & L_2(x_i, x_j) \end{pmatrix} \right)_{i,j=1}^n : M_n(M_2(\mathcal{A})) \rightarrow M_n(M_2(\mathcal{B}))$$

is completely positive.

- (v) Given any finite set $F = \{x_1, x_2, \dots, x_n\}$ of X there exist $L_1, L_2 \in \mathbb{K}_F^+(\mathcal{A}, \mathcal{B})$ such that the map

$$\left(\begin{pmatrix} (L_1(x_i, x_j))_{i,j=1}^n & (k(x_i, x_j))_{i,j=1}^n \\ (k^*(x_i, x_j))_{i,j=1}^n & (L_2(x_i, x_j))_{i,j=1}^n \end{pmatrix} : M_2(M_n(\mathcal{A})) \rightarrow M_2(M_n(\mathcal{B})) \right)$$

is completely positive.

- (vi) Given any finite set $F = \{x_1, x_2, \dots, x_n\}$ of X there exist P_1, P_2 completely positive maps from $M_n(\mathcal{A})$ to $M_n(\mathcal{B})$ such that the map

$$\begin{pmatrix} P_1 & S_{k_F} \\ S_{k_F}^* & P_2 \end{pmatrix} : M_2(M_n(\mathcal{A})) \rightarrow M_2(M_n(\mathcal{B}))$$

is completely positive, where

$$S_{k_F} : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B}) : (a_{i,j})_{i,j=1}^n \mapsto (k(x_i, x_j)[a_{i,j}])_{i,j=1}^n$$

is the Schur product operator associated to the matrix $(k(x_i, x_j))$.

We prove under appropriate conditions that all six statements are in fact equivalent. The assumptions will also be shown to imply the validity of statement (vi), and since Corollary 3.3 gives the equivalence of statement (i) and complete boundedness for a kernel, we arrive at a satisfactory characterisation of such kernels in this setting.

We begin by proving the equivalence of the three “global” statements (i)–(iii), the equivalence of the three “local” statements (iv)–(vi), and that the global statements imply the local statements. The proof draws upon the following two results, the first of which is a routine generalisation of the off-diagonal technique in [11, Theorem 8.3].

NB: Henceforth the C^* -algebra \mathcal{A} is always unital, and the C^* -algebra \mathcal{B} is always injective.

Theorem 4.1. *Let \mathcal{A} be a unital C^* -algebra, and \mathcal{B} be an injective C^* -algebra. Let \mathcal{C} be a unital C^* -subalgebra of both \mathcal{A} and \mathcal{B} . Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a completely bounded, \mathcal{C} -bimodule map. Then there exist completely positive \mathcal{C} -bimodule maps $\varphi_1, \varphi_2 : \mathcal{A} \rightarrow \mathcal{B}$ with $\|\varphi_i\|_{cb} = \|\varphi\|_{cb}$ such that the map*

$$\Phi : M_2(\mathcal{A}) \rightarrow M_2(\mathcal{B}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \varphi_1(a) & \varphi(b) \\ \varphi^*(c) & \varphi_2(d) \end{pmatrix}$$

is completely positive.

Lemma 4.2. *Let $\varphi \in \mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H})))$. Then the following are equivalent.*

- (i) φ is a \mathcal{D}_n -bimodule map.
- (ii) For all $i, j = 1, \dots, n$ and all $A \in M_n(\mathcal{A})$ we have

$$E_{i,j} * \varphi(A) = \varphi(E_{i,j} * A),$$

where $E_{i,j}$ is a matrix unit (that is, the M_n element with 1 in the $(i, j)^{\text{th}}$ position and 0 elsewhere) and $$ is the entrywise (ie, Schur) product.*

- (iii) φ acts entrywise on $M_n(\mathcal{A})$.

Proof. It is clear that (ii) and (iii) of the lemma statement are equivalent. The equivalence of (i) and (ii) follows from

$$E_{i,i} A E_{j,j} = E_{i,j} * A,$$

an easily checked equality. □

Theorem 4.3. *For the above statements, the following implications hold:*

$$(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi).$$

Proof. Statement (ii) is a restatement of (i) using a characterisation of completely positive kernels from [1, Lemma 3.2.1]. The equivalence of (ii) and (iii) follows by using the (complete positivity preserving) canonical shuffle of matrices from [11, Chapter 8], and likewise for (iv) and (v). Statement (iii) implies (v) by a restriction of kernels to a finite subset of X , and statement (v) gives an explicit form for the completely positive maps P_1, P_2 in (vi). It is interesting to note that at this point we have not used the assumptions that \mathcal{A} is unital and \mathcal{B} is injective.

Finally we prove (vi) implies (v). By assumption there exist completely positive maps P_1, P_2 such that

$$\begin{pmatrix} P_1 & S_{k_F} \\ S_{k_F}^* & P_2 \end{pmatrix} \geq 0,$$

so the matrix is a completely bounded map. This implies that S_{k_F} is a completely bounded map. It is a Schur product map, so acts entrywise. By Lemma 4.2 it is a \mathcal{D}_n -bimodule map. Apply Theorem 4.1 and require that the completing maps are \mathcal{D}_n -bimodule maps, P'_1, P'_2 . These are entrywise maps, so can be identified with elements of $M_n(\mathcal{L}(\mathcal{A}, \mathcal{B}))$, which we denote by P''_1, P''_2 . Define completely positive kernels L_1, L_2 on F by

$$L_1(x_i, x_j)[a] := (P''_1)_{i,j}[a], \quad L_2(x_i, x_j)[a] := (P''_2)_{i,j}[a], \quad i, j = 1, \dots, n.$$

These satisfy the conditions of statement (v). \square

5. TOPOLOGIES ON KERNEL SPACES

In this section, we consider topologies on the space $\mathbb{K}_F(\mathcal{A}, \mathcal{B})$ of all kernels on F where F is a finite subset of X . The topologies constructed will be used in the last section to prove, under appropriate restrictions, the existence of Kolmogorov decompositions of semi-uniformly completely bounded kernels when the set X is infinite.

5.1. The topology of bounded linear maps into $\mathcal{L}(\mathcal{H})$. Let \mathcal{A} be a C^* -algebra, \mathcal{H} be a Hilbert space and let $\mathcal{L}^1(\mathcal{H})$ denote the ideal of trace class operators on \mathcal{H} . It is known (cf. [11, Ch 7] for example) that $\mathcal{L}(\mathcal{H}) = \mathcal{L}^1(\mathcal{H})^*$ which allows us to make an identification

$$\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H})) \cong (\mathcal{A} \otimes \mathcal{L}^1(\mathcal{H}))^*.$$

To $\varphi \in \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$ we associate a linear functional L_φ defined (on elementary tensors) by

$$L_\varphi(a \otimes R) = \varphi(a)(R) = \text{tr}(\varphi(a)R).$$

To a linear functional $L \in (\mathcal{A} \otimes \mathcal{L}^1(\mathcal{H}))^*$ and an element $a \in \mathcal{A}$ we associate a linear functional $L^a : \mathcal{L}^1(\mathcal{H}) \rightarrow \mathbb{C} : R \mapsto L(a \otimes R)$. Then we define a bounded, linear map

$$\varphi_L : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}) : a \mapsto L^a.$$

Clearly, $\varphi_{L_\varphi} = \varphi$ and $L_{\varphi_L} = L$. The space $(\mathcal{A} \otimes \mathcal{L}^1(\mathcal{H}))^*$ carries a natural weak-* topology. Thus we endow $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$ with the same topology via the identification above. Formally, we endow $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$ with the weakest topology with respect to which the isometric isomorphism $\varphi \mapsto L_\varphi$ is (weak-*) continuous. This is called the **bounded weak**, or **BW topology**, see [11, Chapter 7].

We define a weaker topology, called the **bounded-bounded weak**, or **BBW topology**, to be the weakest topology with respect to which the evaluations

$$E_{a,R} : \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H})) \rightarrow \mathbb{C} : \varphi \mapsto L_\varphi(a \otimes R)$$

are continuous for all $a \in \mathcal{A}$ and $R \in \mathcal{L}^1(\mathcal{H})$.

Proposition 5.1. *A net φ_α in $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$ converges to φ in the BBW topology if and only if $\varphi_\alpha(a)$ converges weak-* to $\varphi(a)$ for all $a \in \mathcal{A}$.*

Proof. Let φ_α be a net in $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$. Then φ_α converges BBW to φ if and only if $E_{a,R}\varphi_\alpha$ converges to $E_{a,R}\varphi$ for all $a \in \mathcal{A}$ and $R \in \mathcal{L}^1(\mathcal{H})$, if and only if $L_{\varphi_\alpha}(a \otimes R)$ converges to $L_\varphi(a \otimes R)$, if and only if $\varphi_\alpha(a)(R)$ converges to $\varphi(a)(R)$, if and only if $\varphi_\alpha(a)$ converges weak-* to $\varphi(a)$ for all $a \in \mathcal{A}$. \square

We now restate some results from [11] to affirm that useful statements about the bounded weak topology remain true about the bounded-bounded weak topology.

Corollary 5.2. *A bounded net converges BBW if and only if it converges BW.*

Proof. By [11, Lemma 7.2], if φ_α is a bounded net, $\varphi_\alpha(a)$ converges weak-* to $\varphi(a)$ for all $a \in \mathcal{A}$ if and only if φ_α converges BW to φ . \square

Corollary 5.3. *A net φ_α in $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$ converges to φ in the BBW topology if and only if, for all $h, k \in \mathcal{H}$ and $a \in \mathcal{A}$, $\langle \varphi_\alpha(a)h, k \rangle$ converges to $\langle \varphi(a)h, k \rangle$.*

Proof. Combine the previous corollary with [11, Proposition 7.3] \square

Proposition 5.4. *Any bounded, BW-closed subset of $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$ is BBW-closed.*

Proof. Let V be a bounded, BW-closed subset of $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$, and let v be in the BBW-closure of V . Then there is a bounded BBW-convergent net, so a bounded BW-convergent net, converging to v . Hence v is in the BW-closure of the BW-closed set V , i.e. $v \in V$. \square

Proposition 5.5. *Any bounded, BW-compact subset of $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$ is BBW-compact.*

Proof. This is straightforward. Let K be a bounded, BW-compact subset of $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$, and let $\{\mathcal{U}_\lambda\}$ be a BBW-open cover of K . Then $\{\mathcal{U}_\lambda\}$ is a BW-open cover of K , so has a finite subcover, comprising BBW-open sets. \square

Proposition 5.6. *The space $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$ is Hausdorff in the BBW-topology.*

Proof. By virtue of being identified with the continuous linear functionals on $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$, the elements of $\mathcal{A} \otimes \mathcal{L}^1(\mathcal{H})$ separate $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$. So let $\varphi_1 \neq \varphi_2 \in \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$, and suppose that $X \in \mathcal{A} \otimes \mathcal{L}^1(\mathcal{H})$ is such that $L_{\varphi_1}(X) \neq L_{\varphi_2}(X)$. Finite linear combinations of elementary tensor products are norm dense in $\mathcal{A} \otimes \mathcal{L}^1(\mathcal{H})$, so by continuity, we may assume without loss of generality that $X = \sum_j^n a_j \otimes R_j$, where $a_j \in \mathcal{A}$ and $R_j \in \mathcal{L}^1(\mathcal{H})$ for all j . From this it follows that for some elementary tensor $a \otimes R$, $L_{\varphi_1}(a \otimes R) \neq L_{\varphi_2}(a \otimes R)$; that is, $E_{a,R}(\varphi_1) \neq E_{a,R}(\varphi_2)$.

Set $\epsilon = |E_{a,R}(\varphi_1) - E_{a,R}(\varphi_2)|$. For $j = 1, 2$, let B_j the ball of radius $\epsilon/3$ in \mathbb{C} centred at $E_{a,R}(\varphi_j)$ and observe that $E_{a,R}^{-1}(B_j)$ is open by continuity, contains φ_j and $E_{a,R}^{-1}(B_1) \cap E_{a,R}^{-1}(B_2) = \emptyset$, finishing the proof. \square

Write \mathcal{D}_n for the algebra of diagonal, scalar-valued matrices.

Theorem 5.7. *The set $\mathcal{E}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H})))$ of \mathcal{D}_n -bimodule maps in $\mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H})))$ is BBW closed.*

Proof. Let φ_α be a net in $\mathcal{E}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H})))$ converging to $\varphi \in \mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H})))$. We define two maps

$$\psi_1^{i,j} : \mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H}))) \rightarrow \mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H}))) : \theta \mapsto E_{i,j} * \theta,$$

where $(E_{i,j} * \theta)(A) := E_{i,j} * \theta(A)$, and

$$\psi_2^{i,j} : \mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H}))) \rightarrow \mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H}))) : \theta \mapsto \theta * E_{i,j},$$

where $(\theta * E_{i,j})(A) := \theta(E_{i,j} * A)$. It is clear from Lemma 4.2 that elements of $\mathcal{E}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H})))$ are characterised by the property $\psi_1^{i,j}(\theta) = \psi_2^{i,j}(\theta)$ for all $i, j = 1, \dots, n$. Suppose that each of these functions is BBW-continuous. Then, since $\varphi_\alpha \xrightarrow{BBW} \varphi$, it follows that $\psi_1^{i,j}(\varphi_\alpha) \xrightarrow{BBW} \psi_1^{i,j}(\varphi)$ and $\psi_2^{i,j}(\varphi_\alpha) \xrightarrow{BBW} \psi_2^{i,j}(\varphi)$ for all $i, j = 1, \dots, n$. The convergent nets $\psi_1^{i,j}(\varphi_\alpha)$ and $\psi_2^{i,j}(\varphi_\alpha)$ are identical, from which it follows that

$$\psi_1^{i,j}(\varphi) = \psi_2^{i,j}(\varphi), \quad i, j = 1, \dots, n.$$

That is, $\varphi \in \mathcal{E}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H})))$.

It remains to prove that each $\psi_1^{i,j}, \psi_2^{i,j}$ is BBW-continuous. Let us begin by observing that, by virtue of its definition, the weak-* topology on $(M_n(\mathcal{A}) \otimes M_n(\mathcal{L}^1(\mathcal{H})))^*$ is generated by basic open sets of the form

$$\mathcal{U}' = \widehat{(A \otimes R)}^{-1}(\mathcal{B}_\epsilon(z_0)),$$

where $\mathcal{B}_\epsilon(z_0)$ is a ball in \mathbb{C} centred at z_0 of radius $\epsilon > 0$, $A \in M_n(\mathcal{A})$, $R \in \mathcal{L}^1(\mathcal{H}^n) \cong M_n(\mathcal{L}^1(\mathcal{H}))$ and $\widehat{(A \otimes R)}$ is the evaluation function. It follows that the BBW topology has basic open sets of the form

$$\mathcal{U} = \{\varphi \in \mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H}))) : |L_\varphi(A \otimes R) - z_0| < \epsilon\}.$$

Since the weak-* continuous linear functionals separate $(M_n(\mathcal{A}) \otimes M_n(\mathcal{L}^1(\mathcal{H})))^*$, there exists φ_0 such that $L_{\varphi_0}(A \otimes R) \neq 0$. Replacing φ_0 by $\frac{z_0}{L_{\varphi_0}(A \otimes R)} L_{\varphi_0}(A \otimes R)$, we then see that a basis for the BBW topology is generated by open sets

$$\mathcal{U} = \{\varphi \in \mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H}))) : |(L_\varphi - L_{\varphi_0})(A \otimes R)| < \epsilon\}$$

as φ_0, A, R and ϵ vary.

We now consider $\psi_1^{i,j-1}(\mathcal{U})$, for each of these basic open sets. Of course if \mathcal{U} does not intersect the range of $\psi_1^{i,j}$ then $\psi_1^{i,j-1}(\mathcal{U}) = \emptyset$ is trivially open. As above we consider open sets \mathcal{U} centred at $L_{\psi_1^{i,j}(\varphi_0)}(A \otimes R)$ for some φ_0, A and R . Then

$$\psi_1^{i,j-1}\left(\left\{\varphi \in \mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H}))) : \left|(L_\varphi - L_{\psi_1^{i,j}(\varphi_0)})(A \otimes R)\right| < \epsilon\right\}\right)$$

is the set

$$\left\{\varphi \in \mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H}))) : \left|(L_{\psi_1^{i,j}(\varphi)} - L_{\psi_1^{i,j}(\varphi_0)})(A \otimes R)\right| < \epsilon\right\}.$$

Now setting $\psi_1^{i,j}(\theta) = E_{i,j} * \theta$ and $L_\varphi(A \otimes R) = \text{tr}(\varphi(A)R)$ we get that

$$\psi_1^{i,j-1}(\mathcal{U}) = \left\{\varphi \in \mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H}))) : \left|\text{tr}([E_{i,j} * (\varphi - \varphi_0)(A)]R)\right| < \epsilon\right\}.$$

Making the identifications $M_n(\mathcal{L}(\mathcal{H})) = \mathcal{L}(\mathcal{H}^n)$ and $M_n(\mathcal{L}^1(\mathcal{H})) = \mathcal{L}^1(\mathcal{H}^n)$,

$$[E_{i,j} * (\varphi - \varphi_0)(A)]R = [(\varphi - \varphi_0)(A) * E_{i,j}]R = [(\varphi - \varphi_0)(A)][E_{i,j} * R] = [(\varphi - \varphi_0)(A)]\tilde{R}$$

where $\tilde{R} \in \mathcal{L}^1(\mathcal{H}^n)$. Thus

$$\psi_1^{i,j-1}(\mathcal{U}) = \left\{\varphi \in \mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H}))) : \left|\text{tr}((\varphi - \varphi_0)(A)\tilde{R})\right| < \epsilon\right\}$$

is clearly BBW-open.

A substantially identical argument shows that, for \mathcal{U} in the analogously chosen basis,

$$\psi_2^{i,j-1}(\mathcal{U}) = \left\{\varphi \in \mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H}))) : \left|\text{tr}((\varphi - \varphi_0)(E_{i,j} * A)R)\right| < \epsilon\right\}.$$

Defining $\tilde{A} := E_{i,j} * A$, it becomes clear that $\psi_2^{i,j-1}(\mathcal{U})$ is open. Thus $\psi_1^{i,j}$ and $\psi_2^{i,j}$ are continuous, and the result is proven. \square

Finally, some brief observations on relative topologies.

Proposition 5.8. *Let \mathbb{X} be a space, \mathbb{Y} a topological space with topology $\tau_{\mathbb{Y}}$, and let $f : \mathbb{X} \rightarrow \mathbb{Y}$ be injective. Endow \mathbb{X} with the weakest topology $\tau_{\mathbb{X}}$ such that f is continuous. Suppose that $A \subset \mathbb{X}$, so $f(A) \subset \mathbb{Y}$ and denote the relative topologies of A and $f(A)$*

$$\tau_A := \{\mathcal{U} \cap A : \mathcal{U} \in \tau_{\mathbb{X}}\}, \quad \tau_B := \{\mathcal{U} \cap f(A) : \mathcal{U} \in \tau_{\mathbb{Y}}\}.$$

Then τ_A is the weakest topology with respect to which $f|_A$ is (τ_B) -continuous. Further, if $f(A)$ is τ_B -compact, then A is τ_A -compact.

Proof. Let the weakest topology with respect to which $f|_A$ is (τ_B) -continuous be τ'_A . Then

$$\tau'_A := \{f^{-1}(\mathcal{U} \cap f(A)) : \mathcal{U} \in \tau_B\} = \{f^{-1}(\mathcal{U}) \cap A : \mathcal{U} \in \tau_B\} = \tau_A.$$

For compactness, simply note that $f|_A$ is a homeomorphism of (A, τ_A) and $(f(A), \tau_B)$. \square

Lemma 5.9 ([5, §12, Corollary 1, p. 68]). *Let X be a normed, linear space. Then every bounded subset of X^* is relatively weak- $*$ compact.*

Corollary 5.10. *Every bounded subset of $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$ is relatively BBW-compact.*

Proof. Proposition 5.8 tells us that bounded subsets of $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{H}))$ are relatively BW-compact. Let A be such a subset, and take a cover $\{\mathcal{U}_\alpha \cap A\}_\alpha$ for some collection $\{\mathcal{U}_\alpha\}_\alpha$ of BBW-open sets. Since BBW-open sets are BW-open, the sets $\mathcal{U}_\alpha \cap A$ are relatively BW-open. Relative BW-compactness gives the finite subcover we require. \square

5.2. The pointwise σ -BBW topology on the kernels. Let $F = \{x_1, x_2, \dots, x_n\}$, $x_i \in X$. We define the **pointwise σ -BBW topology** τ_F^p on $\mathbb{K}_F(\mathcal{A}, \mathcal{B})$ to be the weakest topology such that for all $x, y \in F$ the maps

$$\mathbb{F}_{x,y} : \mathbb{K}_F(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{B})$$

are continuous, where $\mathcal{L}(\mathcal{A}, \mathcal{B})$ is endowed with the BBW topology. It is then clear that when $G \subset F$, the restriction maps

$$\mathbb{K}_F(\mathcal{A}, \mathcal{B}) \rightarrow \mathbb{K}_G(\mathcal{A}, \mathcal{B}) : k \mapsto k|_G \quad (5.1)$$

are automatically continuous. The evaluations $\mathbb{F}_{x,y}$ separate $\mathbb{K}_F(\mathcal{A}, \mathcal{B})$ and the BBW topology is locally convex and Hausdorff, so each τ_F^p is locally convex and Hausdorff.

5.3. The local σ -BBW topology on the kernels. There is another topology worth considering on kernel spaces. From the previous discussion, for a fixed $F = \{x_1, x_2, \dots, x_n\}$ there is an identification

$$j_F : \mathbb{K}_F(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{L}(M_n(\mathcal{A}), \mathcal{L}(\mathcal{H}^n)) : k \mapsto \widehat{\sigma} \left(S_{(k(x_i, x_j))} \right)$$

of a kernel with the Schur product operator associated to the matrix $(k(x_i, x_j))$, $x_i \in F$. We define the **local σ -BBW topology** τ_F^ℓ on $\mathbb{K}_F(\mathcal{A}, \mathcal{B})$ to be the weakest topology such that j_F is continuous. The map j_F is injective and the BBW topology is a locally convex Hausdorff topology. Hence the local σ -BBW topology is also locally convex and Hausdorff. In general we will abuse notation regarding Schur product operators, writing either $(k(x_i, x_j))_F$ or S_{k_F} for the map $S_{(k(x_i, x_j))}$.

5.4. Equivalence of the two σ -BBW topologies.

Lemma 5.11. *For a given faithful unital $*$ -representation σ of the unital C^* -algebra \mathcal{B} , the pointwise σ -BBW topology on \mathbb{K}_F is the same as the local σ -BBW topology on \mathbb{K}_F .*

Proof. Since the topologies are defined as being the weakest making certain maps continuous, it suffices to show that these maps are continuous in both topologies, and this is done by showing that if a net (k_α) of kernels in \mathbb{K}_F converges to k in one of the topologies, it does so in the other.

So assume that (k_α) is a net of kernels in \mathbb{K}_F converging to k in the local σ -BBW topology τ_F^ℓ . Then for any $\tilde{a} \in M_n(\mathcal{A})$ and $\tilde{R} \in \mathcal{L}^1(\mathcal{H}^n) \cong M_n(\mathcal{L}^1(\mathcal{H}))$,

$$\text{tr} \left(\widehat{\sigma}(k_\alpha(x_i, x_j))[\tilde{a}]\tilde{R} \right) \rightarrow \text{tr} \left(\widehat{\sigma}(k(x_i, x_j))[\tilde{a}]\tilde{R} \right).$$

In particular, if we fix i, j and choose $\tilde{a} = a \otimes E_{ij}$ and $\tilde{R} = R \otimes E_{ji}$, where $a \in \mathcal{A}$, $R \in \mathcal{L}^1(\mathcal{H})$, and E_{ij}, E_{ji} are matrix units in $M_n(\mathbb{C})$, we find that

$$\mathrm{tr} \left(\sigma((k_\alpha(x_i, x_j)[a])R \otimes E_{ii}) \right) \rightarrow \mathrm{tr} \left(\sigma((k(x_i, x_j)[a])R \otimes E_{ii}) \right).$$

Thus $\mathrm{tr} \left(\sigma(k_\alpha(x_i, x_j)[a])R \right) \rightarrow \mathrm{tr} \left(\sigma(k(x_i, x_j)[a])R \right)$, from which it follows that \mathbb{F}_{x_i, x_j} is continuous. The choice of i and j are arbitrary, hence $\tau_F^p \subseteq \tau_F^\ell$.

On the other hand, suppose that (k_α) converges to k in the pointwise σ -BBW topology τ_F^p . Then for $\tilde{R} = (R_{ij}) \in M_n(\mathcal{L}^1(\mathcal{H}))$, $\tilde{a} = (a_{ij}) \in M_n(\mathcal{A})$ and any α ,

$$\mathrm{tr} \left(\hat{\sigma}(k_\alpha(x_i, x_j))[\tilde{a}]\tilde{R} \right) \quad (5.2)$$

is just the sum of the traces of the diagonal elements of $\hat{\sigma}(k_\alpha(x_i, x_j))[\tilde{a}]\tilde{R}$, which are all linear combinations of things of the form $\sigma(k_\alpha(x, y)[a])R$, which by definition of the topology τ_F^p , converge to the same linear combinations, with k_α replaced by k . Consequently, (5.2) converges to $\mathrm{tr} \left(\hat{\sigma}(k(x_i, x_j))[\tilde{a}]\tilde{R} \right)$. We conclude that $\tau_F^\ell \subseteq \tau_F^p$, and therefore the two topologies agree. \square

Henceforth we denote these two equivalent topologies on $\mathbb{K}_F(\mathcal{A}, \mathcal{B})$ by τ_F . We have already observed that the restriction maps in (5.1) are continuous when $\mathbb{K}_F(\mathcal{A}, \mathcal{B})$ has the τ_F topology. We endow a ball of completely bounded kernels

$$\mathbb{K}_F^r(\mathcal{A}, \mathcal{B}) := \left\{ k \in \mathbb{K}_F(\mathcal{A}, \mathcal{B}) : \left\| (k(x_i, x_j))_F \right\|_{cb} \leq r \right\}$$

with a relative τ_F -topology, denoted τ_F^r .

Lemma 5.12. *Let \mathcal{B} be unital. The balls*

$$\mathbb{K}_F^r(\mathcal{A}, \mathcal{B}) := \left\{ k \in \mathbb{K}_F(\mathcal{A}, \mathcal{B}) : \left\| (k(x_i, x_j))_F \right\|_{cb} \leq r \right\}$$

are τ_F^r -compact.

Proof. The set $j_F(\mathbb{K}_F^r(\mathcal{A}, \mathcal{B})) = j_F(\mathbb{K}_F(\mathcal{A}, \mathcal{B})) \cap \mathrm{CB}_F^r$ is bounded, so by Corollary 5.10 it is relatively BBW-compact. Endow $\mathbb{K}_F^r(\mathcal{A}, \mathcal{B})$ with the weakest topology such that the restricted map

$$j_F^r : \mathbb{K}_F^r(\mathcal{A}, \mathcal{B}) \rightarrow j_F(\mathbb{K}_F^r(\mathcal{A}, \mathcal{B})) : k \mapsto j_F(k)$$

is continuous. By Proposition 5.8, that topology is τ_F^r , j_F^r is a homeomorphism, and $\mathbb{K}_F^r(\mathcal{A}, \mathcal{B})$ is compact in that topology. \square

Proposition 5.13. *The truncated positive cone*

$$\mathbb{K}_F^{r,+}(\mathcal{A}, \mathcal{B}) := \left\{ k \in \mathbb{K}_F^+(\mathcal{A}, \mathcal{B}) : \left\| (k(x_i, x_j))_F \right\|_{cb} \leq r \right\}$$

is τ_F^r -compact.

Proof. By [11, Proposition 7.4] the truncated cone $CP^r(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H})))$ of positive elements in $\mathcal{L}(M_n(\mathcal{A}), M_n(\mathcal{L}(\mathcal{H})))$ is closed. The map j_F^r is continuous, so the pre-image of that cone, the truncated cone of kernels, is τ_F^r -closed in the τ_F^r -compact ball of completely bounded kernels, so is itself τ_F^r -compact. \square

Now we obtain a characterisation of convergence in τ_F^r in the manner of [11, Proposition 7.3], or Corollary 5.3.

Lemma 5.14. *Let \mathbb{X} be a topological space, and $A \subset \mathbb{X}$ a closed subset. Then a net $\{a_\alpha\} \subset A$ converges to $a \in A$ in the relative topology if and only if $a_\alpha \rightarrow a$ in \mathbb{X} .*

Proof. Suppose $a_\alpha \rightarrow a$ in the relative topology. Take an open neighbourhood \mathcal{U} of a , in \mathbb{X} , so $A \cap \mathcal{U}$ is a relatively open neighbourhood of a in A . Then by our supposition a_α is eventually in $A \cap \mathcal{U}$, so is eventually in \mathcal{U} . Therefore $a_\alpha \rightarrow a$ in \mathbb{X} . Conversely, let $a_\alpha \rightarrow a$ in \mathbb{X} . Since A is closed, $a \in A$. Take a relatively open neighbourhood of a , $A \cap \mathcal{U}$, \mathcal{U} open in \mathbb{X} . Then a_α is always in A and eventually in \mathcal{U} , so is eventually in $A \cap \mathcal{U}$. Thus $a_\alpha \rightarrow a$ relatively. \square

Proposition 5.15. *Let k_α be a net in $\mathbb{K}_F^r(\mathcal{A}, \mathcal{B})$. Let $\sigma : \mathcal{B} \rightarrow \mathcal{L}(\mathcal{H})$ be a $*$ -representation, and allow σ to induce j_F, j_F^r and τ_F^r as described previously. Then $k_\alpha \rightarrow k$ in τ_F^r if and only if for all $(a_{i,j}) \in M_n(\mathcal{A})$, $\{h_i\}_{i=1}^n, \{k_i\}_{i=1}^n \subset \mathcal{H}$ we have*

$$\langle \sigma^{(n)}((k_\alpha(x_i, x_j))_F(a_{i,j}))(\oplus_i h_i), \oplus_i k_i \rangle \rightarrow \langle \sigma^{(n)}((k(x_i, x_j))_F(a_{i,j}))(\oplus_i h_i), \oplus_i k_i \rangle.$$

Proof. Since j_F^r is a homeomorphism, $k_\alpha \xrightarrow{\tau_F^r} k$ if and only if $j_F^r(k_\alpha) \rightarrow j_F^r(k)$ in the relative BBW topology. By Lemma 5.14, this is equivalent to $j_F(k_\alpha) \rightarrow j_F(k)$ in the BBW topology. The proof then follows from Corollary 5.3 and the definition of j_F . \square

6. CHARACTERISATION OF COMPLETELY BOUNDED KERNELS

Well order the domain X and form a directed set Λ of finite strictly increasing sequences $\{\lambda_i\}$ of elements of X . We say $\alpha \leq \beta$ in Λ whenever α is a subsequence of β (i.e. there exists an injective, order-preserving map

$$\tilde{\alpha} : \{1, 2, \dots, |\alpha|\} \rightarrow \beta$$

such that $\tilde{\alpha}(j) = \alpha_j$).

Let $k \in \mathbb{K}_X(\mathcal{A}, \mathcal{B})$ be a completely bounded kernel, $F \in \Lambda$. For each such $F = \{x_1, \dots, x_n\}$, we can define a kernel $k_F \in \mathbb{K}_F(\mathcal{A}, \mathcal{B})$ by

$$k_F(x, y) = k(x, y), \quad x, y \in F.$$

Obviously k_F is a completely bounded kernel, and so by definition $(k(x_i, x_j))_{i,j=1}^n : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ is a completely bounded map. Hence by [11, Theorem 8.3], there exist completely positive maps $P_1, P_2 : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ such that statement (vi) of Section 4 holds. Theorem 4.3 then gives the equivalence of statements (vi) and (iv), and it then follows by Corollary 3.3 that k_F has a Kolmogorov decomposition. Recall that this means that there exists a triple $(\tilde{E}_F, \tilde{J}_F, \tilde{\iota}_F)$ such that \tilde{E}_F is an $(\mathcal{A}, \mathcal{B})$ -correspondence, $J_F \in \mathcal{L}^a(E_F)$ is a contractive left \mathcal{A} -module map and $\tilde{\iota}_F : F \rightarrow \tilde{E}_F$ such that for all choices of x, y, a we have:

$$k_F(x, y)[a] = \langle a \cdot \tilde{\iota}_F(y), \tilde{J}_F(\tilde{\iota}_F(x)) \rangle.$$

Next define kernels $\varphi_F \in \mathbb{K}_X(\mathcal{A}, \mathcal{B})$ by

$$\varphi_F(x, y) = \begin{cases} k_F(x, y) & x, y \in F; \\ 0 & \text{otherwise.} \end{cases}$$

If we set $E_F = \tilde{E}_F$, $J_F = \tilde{J}_F$, and define $\iota_F : X \rightarrow E_k$ by

$$\iota_F(x) = \begin{cases} \tilde{\iota}_F(x) & x \in F; \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\varphi_F(x, y)[a] = \langle a \cdot \iota_F(y), J_F(\iota_F(x)) \rangle$$

is a Kolmogorov decomposition of φ_F . Combining Corollary 3.3 and (i) implies (vi) of Theorem 4.3, we conclude that φ_F is completely bounded.

It is clear that finite linear combinations of kernels in $\mathbb{K}_X(\mathcal{A}, \mathcal{B})$ are also in $\mathbb{K}_X(\mathcal{A}, \mathcal{B})$, and in particular then $\varphi := \sum_F \alpha_F \varphi_F \in \mathbb{K}_X(\mathcal{A}, \mathcal{B})$ when the sum over a finite collection of $F \in \Lambda$ with $\alpha_F \in \mathbb{C}$. The kernel φ so defined is 0 off of a finite set, and by the same argument as above, φ has a Kolmogorov decomposition.

With this in mind, define for $(x, y) \in X \times X$,

$$\tilde{\varphi}_{(x,y)} = \begin{cases} \varphi_{\{x\}} & x = y; \\ \tilde{\varphi}_{(x,y)} = \varphi_{\{x,y\}} - \varphi_{\{x\}} - \varphi_{\{y\}} & x \neq y. \end{cases}$$

Observe that in order that $\tilde{\varphi}_{(x,y)}(x', y') \neq 0$, either $x' = x$ and $y' = y$ or $x' = y$ and $y' = x$, in which case $\tilde{\varphi}_{(x,y)}(x, y) = k(x, y)$ and $\tilde{\varphi}_{(x,y)}(y, x) = k(y, x)$. Hence $\tilde{\varphi}_{(x,y)}$ and $\tilde{\varphi}_{(w,z)}$ are nonzero in the same places if and only if either $w = x$ and $z = y$ or $w = y$ and $z = x$, in which case the two kernels are the same.

Write $(E_{(x,y)}, J_{(x,y)}, \iota_{(x,y)})$ for the triple used in defining the Kolmogorov decomposition of $\tilde{\varphi}_{(x,y)}$. Observe that since $\tilde{\varphi}_{(x,y)}$ is zero off of the set $\{x, y\}$, we may without loss of generality take $\iota_{(x,y)}(x') = 0$ if $x' \notin \{x, y\}$. Recall from the statement and proof Lemma 3.2 that the existence of a Kolmogorov decomposition for $\tilde{\varphi}_{(x,y)}$ is equivalent to the existence of a positive kernels $L_{(x,y)}$ such that

$$(x', y') \mapsto \begin{pmatrix} L_{(x,y)}(x', y') & \tilde{\varphi}_{(x,y)}(x', y') \\ \tilde{\varphi}_{(x,y)}^*(x', y') & L_{(x,y)}(x', y') \end{pmatrix} : \mathcal{A} \rightarrow M_2(\mathcal{B})$$

defines a completely positive kernel, where

$$L_{(x,y)}(x', y')[a] := \langle a \cdot \iota_{(x,y)}(y'), \iota_{(x,y)}(x') \rangle.$$

One sees that $L_{(x,y)}(x', y') = 0$ unless $x', y' \in \{x, y\}$.

For any $F \in \Lambda$, k_F is the restriction to $F \times F$ of

$$\frac{1}{2} \left[\sum_{(x,y) \in F \times F} \tilde{\varphi}_{(x,y)} + \sum_{x \in F} \tilde{\varphi}_{(x,x)} \right],$$

and consequently, for the completely positive kernel L_F^0 defined as the restriction to $F \times F$ of

$$\frac{1}{2} \left[\sum_{(x,y) \in F \times F} L_{(x,y)} + \sum_{x \in F} L_{(x,x)} \right],$$

we have that

$$\begin{pmatrix} L_F^0 & k_F \\ k_F^* & L_F^0 \end{pmatrix}$$

defines a completely positive kernel. Note in particular that by definition, for $G \subset F$, $L_F^0|_{G \times G} = L_G^0$.

For each $F \in \Lambda$, we set

$$r_F := \left\| S_{L_F^0} \right\|_{cb}$$

and inductively define the space of local solutions as

$$\mathbb{L}_F := \left\{ (L_1, L_2) \in \mathbb{K}_F^{r_F, +}(\mathcal{A}, \mathcal{B}) \times \mathbb{K}_F^{r_F, +}(\mathcal{A}, \mathcal{B}) : \begin{pmatrix} L_1(x_i, x_j) & k(x_i, x_j) \\ k^*(x_i, x_j) & L_2(x_i, x_j) \end{pmatrix} \geq 0 \right. \\ \left. \text{and } (L_1|_{G \times G}, L_2|_{G \times G}) \in \mathbb{L}_G \text{ for all } G \leq F \right\}.$$

This set is non-empty, since it contains (L_F^0, L_F^0) .

The truncated positive cone $\mathbb{K}_F^{r_F,+}(\mathcal{A}, \mathcal{B})$ is, by Proposition 5.13, $\tau_F^{r_F}$ -compact, and so $\mathbb{K}_F^{r_F,+}(\mathcal{A}, \mathcal{B}) \times \mathbb{K}_F^{r_F,+}(\mathcal{A}, \mathcal{B})$ is compact in the product topology $\tau_F^{r_F} \times \tau_F^{r_F}$. The space is also Hausdorff. Thus, to show that \mathbb{L}_F is $\tau_F^{r_F} \times \tau_F^{r_F}$ -compact requires only that it be closed. We do this inductively. If F is a one point set, the result is clearly true. So assume the cardinality of F , $|F| > 1$ and that we know that \mathbb{L}_G is $\tau_G^{r_G}$ -compact for any set G with $|G| < |F|$. Let (L_1^α, L_2^α) be a convergent net in \mathbb{L}_F . By compactness of the positive cones, its limit is (L_1, L_2) where L_1, L_2 are completely positive kernels, with $\|(L_1(x_i, x_j))\|_{cb}, \|(L_2(x_i, x_j))\|_{cb} \leq r_F$. Furthermore, by definition of \mathbb{L}_F , the induction hypothesis, and the fact that, since $r_G \leq r_F$, the $\tau_G^{r_G}$ topology on $\mathbb{K}_G^{r_G,+}(\mathcal{A}, \mathcal{B})$ is the restriction of the $\tau_F^{r_F}$ topology from $\mathbb{K}_F^{r_F,+}(\mathcal{A}, \mathcal{B})$, the net $(L_1^\alpha|_{G \times G}, L_2^\alpha|_{G \times G})$ converges to $(L_1|_{G \times G}, L_2|_{G \times G}) \in \mathbb{L}_G$ for $G < F$.

By Proposition 5.15, $L^\alpha \xrightarrow{\tau_F^{r_F}} L$ if and only if for all $(a_{i,j}) \in M_n(\mathcal{A})$, $\{h_i\}_{i=1}^n, \{k_i\}_{i=1}^n \subset \mathcal{H}$ we have

$$\langle \sigma^{(n)}((L_\alpha(x_i, x_j))_F(a_{i,j}))(\oplus_i h_i), \oplus_i k_i \rangle \rightarrow \langle \sigma^{(n)}((L(x_i, x_j))_F(a_{i,j}))(\oplus_i h_i), \oplus_i k_i \rangle.$$

We regard the space of 2×2 matrices with entries from $\mathbb{K}_F^{r_F}(\mathcal{A}, \mathcal{B})$ as sitting inside $\mathbb{K}_F^{4r_F}(\mathcal{A}, M_2(\mathcal{B}))$ and endow that set with a relative topology, like $\tau_F^{r_F}$, but now induced by the representation $\sigma_2 : M_2(\mathcal{B}) \rightarrow \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. Call this topology τ . As we have shown (cf. Proposition 5.13) the set of positive elements is τ -closed, and we have a characterisation of convergence (cf. Proposition 5.15).

Since $(L_1^\alpha, L_2^\alpha) \rightarrow (L_1, L_2)$, for all choices of $(a_{i,j}) \in M_n(\mathcal{A})$ and $(h'_i), (k'_i), (h''_i), (k''_i) \in \oplus_{i=1}^n \mathcal{H}$ as above, we have

$$\begin{aligned} & \langle \sigma^{(n)}((L_1^\alpha(x_i, x_j))[a_{i,j}])(\oplus_i h'_i), \oplus_i k'_i \rangle + \langle \sigma^{(n)}((L_2^\alpha(x_i, x_j))[a_{i,j}])(\oplus_i h''_i), \oplus_i k''_i \rangle \\ & \rightarrow \langle \sigma^{(n)}((L_1(x_i, x_j))[a_{i,j}])(\oplus_i h'_i), \oplus_i k'_i \rangle + \langle \sigma^{(n)}((L_2(x_i, x_j))[a_{i,j}])(\oplus_i h''_i), \oplus_i k''_i \rangle \end{aligned}$$

We then add fixed terms to both sides,

$$\begin{aligned} & \langle \sigma^{(n)}((L_1^\alpha(x_i, x_j))[a_{i,j}])(\oplus_i h'_i), \oplus_i k'_i \rangle + \langle \sigma^{(n)}((L_2^\alpha(x_i, x_j))[a_{i,j}])(\oplus_i h''_i), \oplus_i k''_i \rangle \\ & + \langle \sigma^{(n)}((k(x_i, x_j))[a_{i,j}])(\oplus_i h'_i), \oplus_i k'_i \rangle + \langle \sigma^{(n)}((k^*(x_i, x_j))[a_{i,j}])(\oplus_i h'_i), \oplus_i k'_i \rangle \\ & \rightarrow \langle \sigma^{(n)}((L_1(x_i, x_j))[a_{i,j}])(\oplus_i h'_i), \oplus_i k'_i \rangle + \langle \sigma^{(n)}((L_2(x_i, x_j))[a_{i,j}])(\oplus_i h''_i), \oplus_i k''_i \rangle \\ & + \langle \sigma^{(n)}((k(x_i, x_j))[a_{i,j}])(\oplus_i h'_i), \oplus_i k'_i \rangle + \langle \sigma^{(n)}((k^*(x_i, x_j))[a_{i,j}])(\oplus_i h'_i), \oplus_i k'_i \rangle \end{aligned}$$

and rewrite this as

$$\begin{aligned} & \left\langle \begin{pmatrix} \sigma^{(n)}((L_1^\alpha(x_i, x_j))[a_{i,j}]) & \sigma^{(n)}((k(x_i, x_j))[a_{i,j}]) \\ \sigma^{(n)}((k^*(x_i, x_j))[a_{i,j}]) & \sigma^{(n)}((L_2^\alpha(x_i, x_j))[a_{i,j}]) \end{pmatrix} \begin{pmatrix} \oplus_{i=1}^n h'_i \\ \oplus_{i=1}^n h''_i \end{pmatrix}, \begin{pmatrix} \oplus_{i=1}^n k'_i \\ \oplus_{i=1}^n k''_i \end{pmatrix} \right\rangle \\ & \rightarrow \left\langle \begin{pmatrix} \sigma^{(n)}((L_1(x_i, x_j))[a_{i,j}]) & \sigma^{(n)}((k(x_i, x_j))[a_{i,j}]) \\ \sigma^{(n)}((k^*(x_i, x_j))[a_{i,j}]) & \sigma^{(n)}((L_2(x_i, x_j))[a_{i,j}]) \end{pmatrix} \begin{pmatrix} \oplus_{i=1}^n h'_i \\ \oplus_{i=1}^n h''_i \end{pmatrix}, \begin{pmatrix} \oplus_{i=1}^n k'_i \\ \oplus_{i=1}^n k''_i \end{pmatrix} \right\rangle. \end{aligned}$$

Perform a canonical shuffle on the large matrix, and relabel the Hilbert space elements, to conclude that for all choices of $(a_{i,j}), (h_i)$ and (k_i) ,

$$\begin{aligned} & \left\langle \sigma^{(2n)} \left(\begin{pmatrix} L_1^\alpha(x_i, x_j) & k(x_i, x_j) \\ k^*(x_i, x_j) & L_2^\alpha(x_i, x_j) \end{pmatrix} [(a_{i,j})] \right) (\oplus_{i=1}^{2n} h_i), \oplus_{i=1}^{2n} k_i \right\rangle \\ & \rightarrow \left\langle \sigma^{(2n)} \left(\begin{pmatrix} L_1(x_i, x_j) & k(x_i, x_j) \\ k^*(x_i, x_j) & L_2(x_i, x_j) \end{pmatrix} [(a_{i,j})] \right) (\oplus_{i=1}^{2n} h_i), \oplus_{i=1}^{2n} k_i \right\rangle \end{aligned}$$

It follows from Proposition 5.15 that

$$\begin{pmatrix} L_1^\alpha & k|_F \\ k|_F^* & L_2^\alpha \end{pmatrix} \xrightarrow{\tau} \begin{pmatrix} L_1 & k|_F \\ k|_F^* & L_2 \end{pmatrix}. \quad (6.1)$$

The net on the left of (6.1) is composed of positive elements, and the positive elements are τ -closed, so the limit is a completely positive kernel. Consequently, (L_1, L_2) belongs to the set of solutions \mathbb{L}_F , and so \mathbb{L}_F is a non-empty compact Hausdorff space for each F . Also, by definition of $\tau_G^{r_G}$ and $\tau_F^{r_F}$, for $G \leq F$ in Λ the restriction maps $f_{G,F} : \mathbb{L}_F \rightarrow \mathbb{L}_G : L \mapsto L|_G$ are automatically continuous.

We have constructed an *inverse limit system* of non-empty compacta, indexed by a directed set Λ and connected by continuous maps (restrictions) f_{λ_1, λ_2} for $\lambda_1 \leq \lambda_2$, such that $f_{\lambda, \lambda}$ is the identity and $f_{\lambda_1, \lambda_2} f_{\lambda_2, \lambda_3} = f_{\lambda_1, \lambda_3}$ when $\lambda_1 \leq \lambda_2 \leq \lambda_3$ (cf. for example, [2, Theorem 6.B.11]). We conclude that the inverse limit system is non-empty: that is, there exists

$$((L_1^\lambda, L_2^\lambda)) \in \prod_{\lambda \in \Lambda} \mathbb{L}_\lambda \quad \text{and} \quad f_{\lambda_1, \lambda_2}((L_1^{\lambda_2}, L_2^{\lambda_2})) = (L_1^{\lambda_1}, L_2^{\lambda_1}) \quad \text{whenever } \lambda_1 \leq \lambda_2.$$

It follows that this object uniquely specifies an element of $\mathbb{K}_X^+(\mathcal{A}, \mathcal{B})$ satisfying statement (i).

The following summarises what we have learned.

Theorem 6.1. *Let \mathcal{A} be a unital C^* -algebra and \mathcal{B} be an injective C^* -algebra. Let $k \in \mathbb{K}_X(\mathcal{A}, \mathcal{B})$ be a completely bounded kernel. Then k has a Kolmogorov decomposition, by which we mean a triple (E_k, J, ι) such that E_k is an $(\mathcal{A}, \mathcal{B})$ -correspondence, $J \in \mathcal{L}^a(E_k)$ is a contractive left \mathcal{A} -module map and $\iota : X \rightarrow E_k$ such that for all choices of x, y, a we have:*

$$k(x, y)[a] = \langle a \cdot \iota(y), J(\iota(x)) \rangle.$$

Furthermore, k is:

- (i) *completely positive if and only if $J \geq 0$,*
- (ii) *hermitian if and only if $J = J^*$,*

and k can be expressed as a linear combination of at most four completely positive kernels.

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